THE

MATHEMATICAL GAZETTE.

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.; PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.;
PROF. E. T. WHITTAKER, M.A., F.R.S.

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AN INTERESTING FIND.

THE letter of which a reduced *facsimile* is here given was lately discovered bound up with a set of pamphlets unconnected with it in any way. It extends to $2\frac{3}{4}$ pages of about double the page of type in the *Gazette*, the matter on the last page being selected for reproduction as most interesting, not only on account of the very clear signature, but on that of the paragraphs bearing on mathematical studies. The scale of the reproduction is about one-half that of the original. The views expressed on the application of algebra to geometry will probably be unexpected by most readers, as only four years later than the date of the letter (1711) Taylor published his *Methodus Incrementorum*, and in it his now famous theorem. Though the latter has made his name more widely known than any other of his works, it is not inopportune to mention the influence he has had on the scientific treatment of perspective in his two works: (1) *Linear Perspective* (1715), or a new method of representing justly all manner of objects as they appear to the eye in all situations; (2) *New Principles of Linear Perspective* (1719), or the art of designing on a plane the representation of all sorts of objects in a more general and simple method than has been done before.

The first is mathematical, the second intended for artists not versed in geometry. Of the statement that Taylor took his use of vanishing points from Obaldi's *Perspectivæ Libri Sex* (1600), De Morgan remarks that whereas Obaldi only introduced the use of them as to lines which are horizontal, Taylor applied them to all lines whatsoever.

Returning to the letter, about two-thirds of it is devoted to the investigation of the path of a ray of light through a medium of gradually increasing density. He then passes to the recent election of Saunderson to the Lucasian professorship just made vacant by the deprivation of Whiston. The Rev. Mr. Newcome, to whom he writes, would seem to have had the professorship offered to him, and to have declined it. Expressing his regret at this, he goes on to say: "Since it is so, I am very glad Mr. Saunderson has it, and hope he will fully answer the expectations the electors have of him. He is an extraordinary algebraist, and I expect great improvements from his hand," etc. (See the reproduction.)

The faint flourish which partly encircles "yr. most sincere" is an accidental reproduction by the photographic plate of part of the address—no envelope

being used—written on the fourth page of the sheet. The tearing is apparently due to the breaking of the seal when the letter was opened.

EDWARD M. LANGLEY.

Can any reader of the *Gazette* identify "Dear Tom"? A Newcome was elected master of St. John's in 1735, but his name is given in the calendar as *John* Newcome.

DIVISION OF ANGLES AND ARCS.

THE rule which is usually given in works on Geometrical Drawing for constructing a regular polygon of n sides is as follows: *Take a circle with diameter AB. On AB describe an equilateral triangle ABC. Divide AB into n equal parts. Let Q be the second point of section reckoning from A. If CQ cut the circle at P, then AP will be the n^{th} part of the circumference.*

The accuracy of this rule must be that of the theorem that *if ABC be an equilateral triangle described on the base AB of a semicircle, then straight lines through C divide the base and the curve in the same ratio.*

To verify this proposition, consider a straight line through C which divides the base at Q in the ratio $90-x : 90+x$ and the curve at P in the ratio $90-y : 90+y$ so that the angle DOP is y° and the ratio $OQ : OA$ is $x : 90$.

The relation between x and y is

$$x = 90\,\frac{\sqrt{3}\sin y^\circ}{\sqrt{3}+\cos y^\circ}.$$

It is evident that this equation reduces to $x=y$ in five cases, viz. $y=0$, ± 30, ± 90. Calculation shews that the statement is not far from the truth for intermediate values of y. The table of corresponding values of x and y calculated for every 5° is worth examination.

y	0	5	10	15	20	25	30	35	40	45
x	0	5−	10−	15−	20−	25−	30	35+	40+	45+
$x-y$	0	−·0212	−·0366	−·0457	−·0446	−·0301	0	·0470	·1108	·1906
y	50	55	60	65	70	75	80	85	90	
x	50+	55+	60+	65+	70+	75+	80+	85+	90	
$x-y$	·2832	·3832	·4837	·5689	·6258	·6317	·5564	·3594	0	

In the graph Fig. III. the scale for $x-y$ ordinates is 100 times the scale for y abscissae.

It will be seen that $x-y$ does not exceed ·65 at any point, and that between 0° and 30° this error is less than ·05°. The accuracy attainable in Geometrical Drawing is not closer than two tenths of a degree, so the theorem which was enunciated above is proved to be accurate for all practical purposes when the ratio of the segments lies between 1 : 2 and 2 : 1, but the error may be appreciable when the ratio lies outside these limits.

It appears from the table that if we were to apply the conventional rule to the construction of the regular 20-gon, the error in the arc of 18° (72° in the table) would be as much as ·63°. Multiplying the error by 20, we find

that the 20 arcs would exceed a complete circumference by 18°. If on the other hand we were to use the arc of 18° measured from the middle of the arc AB, then the error in one arc would be ·045 and the 20 arcs would not fall short of a complete circumference by more than 1°.

This comparison leads us to suggest as an improved form of the rule for constructing the polygon of n sides when $n > 6$ viz.: Take a circle with centre O and diameter AB. On AB describe an equilateral triangle ABC. Divide OA at Q in the ratio $4 : n \sim 4$. Produce CO, CQ to meet the circumference. The arc they intercept will be one n^{th} of the circumference.

The problem, to divide an angle into any number of parts, is reduced by the theorem of proportionate division to dividing a finite straight line into the same number of parts. The figure which is drawn to shew the trisection of an angle, shews how simple the method is. The error in theory is quite negligible compared with the inevitable error in draughtsmanship. The method of trial and error remains the only practically useful one for all questions of this type.

T. J. W. WHIPPLE.

CONTRACTED MULTIPLICATION.

I HAVE read with interest the Note on this subject in the January number of the Gazette, and the report of the discussion at the Meeting of the Association. Mr. Lodge suggests figuring the multiplicand. It seems to me, however, less cumbersome to reverse the multiplier at the start. Very fair results have been obtained at Bromsgrove School by the method of multiplication indicated below. Written out at length, the method seems at first sight rather complicated, but in actual practice it is easy to learn.

The method is as follows:

(1) Arrange the multiplier with its highest figure in the units place.

(2) Cut off from the multiplicand, by a vertical line, the number of figures to be obtained correct.

(3) Place the units figure of the multiplier on the right of the line and write the latter in reverse order.

(4) Each digit of the multiplier is now below the digit of the multiplicand which gives the first figure of the partial product retained. Look, however, one place to the right, and carry from it to the nearest ten.

(5) Write the first figure of each partial product in the column on the right of the line, and in adding up this column carry to the nearest ten and put nothing down.

Examples.

I. Find the value of $85{\cdot}362 \times {\cdot}04796$ to the nearest hundredth.

Product $= {\cdot}85362 \times 4{\cdot}796$.

·85 362		
6 97 4		
3 41 4	4 × 6, 24	carry 2, 4 × 3, 12, 14.
59 7	7 × 3, 21	„ 2, 7 × 5, 53, 37.
7 7	9 × 5, 45	„ 5, 9 × 8, 72, 77.
5	6 × 8, 48	„ 5.
4·09	Answer.	

Rough Check $85 \times {\cdot}05 = 4{\cdot}25$.

The decimal point in the result is always under that in the top line.

II. Find the value of $4{\cdot}785 \times 23{\cdot}68 \times {\cdot}09864$ to the nearest tenth.

Product $= 4{\cdot}785 \times 2{\cdot}368 \times {\cdot}9865.$

$$
\begin{array}{r|l}
 & {\cdot}9865 \\
 & 8632 \\
\hline
 & 19730 \\
 & 2960 \\
 & 592 \\
 & 78 \\
\hline
 & 2{\cdot}3636 \\
58 & 74 \\
\hline
 & 934 \\
 & 163 \\
 & 18 \\
 & 1 \\
\hline
 & 11{\cdot}2
\end{array}
$$

Answer.

Rough Check $4{\cdot}8 \times 24 \times {\cdot}1 = 11{\cdot}52$ or more shortly $5 \times 20 \times {\cdot}1 = 10.$

The first multiplication is always carried two figures further than the final result.

III. The length and breadth of a rectangular field are known to be 15·34 chains and 9·63 chains, correct to the nearest link. Find the average of the field to the degree of accuracy obtainable.

$$
\left.
\begin{aligned}
\text{Greatest possible error} &= (15{\cdot}34 \times 9{\cdot}63) \times {\cdot}005 \\
&= 25 \times {\cdot}005 \\
&= {\cdot}125
\end{aligned}
\right\} \text{roughly.}
$$

We can obtain the answer correct to the nearest square chain.

$$
\begin{array}{r|l}
15{\cdot} & 34 \\
36 & 9 \\
\hline
138 & 1 \\
9 & 2 \\
 & 5 \\
\hline
148 & \text{sq. ch.}
\end{array}
$$

14·8 acres. Answer.

The examples given above do not contain a large number of figures, but they illustrate the method sufficiently. In this brief note I have not attempted to justify the method, but any Arithmetician will be able to obtain the reasons for the various steps by writing out the full multiplication for Example I. and comparing it with the contracted solution. Any teacher will do this as a matter of course when he introduces the subject.

FRANK CLARKE.

IS THERE NEED OF A RECOGNIZED SEQUENCE IN GEOMETRY?

I AM glad that Mr. Child has raised this question. We have had some years of liberty in geometry teaching; we ought by this time to have made up our minds how we like it.

To begin with, I doubt if there is any justification for speaking of the "Cambridge sequence." Cambridge has issued a standard list of theorems; and these theorems were necessarily printed in a certain order. But I understand there was no intention of recommending this order in preference to any other. Certainly there is no internal evidence, in the schedule, of any such intention. Many of us will remember Prof. Forsyth's speech at the General Meeting of the M.A. two or three years ago. A proposal had been brought forward to fix a sequence. On that occasion Prof. Forsyth strongly deprecated any such attempt to return to bondage. I understand that his views have not changed in the meantime.

For my own part, I still agree with the views that Prof. Forsyth expressed on that occasion. I have not found, either as teacher or examiner, that the variety of text-book and sequence causes any real inconvenience.

If the inconvenience is ever felt, it will be when a boy comes from his preparatory to his public school. There will always be a slight difficulty in teaching a class of new boys, whatever the subject. Perhaps no two of them have been taught alike. The difficulty (such as it is) is perennial and inevitable; but no one thinks of making a fuss about it, except in geometry. I do not believe that there is really any peculiar difficulty, due to this cause, in teaching Geometry.

One expects the boys to know *the facts* of a certain number of theorems, in order that they may solve riders. Commonly they do not know these facts, and this is where the trouble occurs. Provided they know the facts, however, I do not mind what proof they have learnt. I do not even mind if they are unable to prove the theorems; there will be plenty of time for them to learn the proofs before they have to pass Littlego, Responsions—or what not; in the meantime, it will be a nice exercise for them to rediscover the proofs, as original riders.

When the evil day of the Littlego draws near, then will be the time to make sure of the proofs, and to make sure of the order in which the theorems are arranged. Provided that their Geometry work has been on the right lines, with plenty of riders and a reasonable amount of drawing, there will be no difficulty whatever with this revision of the modest list of propositions required now-a-days. This remark is not mere optimism: it is founded on experience.

As to the order in which to take the four subjects, namely Book I., areas, circle, similarity, there are various permutations and there is room for much pretty discussion. My private belief is that it does not matter greatly which of the competing orders is adopted.

If one is anxious to bring in mathematics early in connection with mensuration and laboratory work, there is something to be said for putting areas early. Areas are much more useful than circle or similarity properties in this work.

Again, the circle provides a wealth of easy riders; whereas it is rather difficult to find interesting riders on "similarity."

These and similar considerations, together with a conservative instinct, have led me to prefer the old-fashioned Euclidean order for the four main subjects. Other people for equally good reasons may prefer another order, and will have no difficulty in finding a suitable text-book. Here again the policy of freedom seems to be vastly preferable to any kind of orthodoxy. New boys at public schools have seldom done more than Book I.; so that this particular difficulty does not enter at this stage. C. GODFREY.

ANSWERS TO QUERIES.

1. [I. 2. b.] *No. of figures expressing* $\frac{1}{N}$ *in scale r.*

r	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
N																			
2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	2	1	1	2	1	1	2	1	1	2	1	1	2	1	1	2	1	1	2
4	2	2	1	1	2	2	1	1	2	2	1	1	2	2	1	1	2	2	1
5	4	4	2	1	1	4	4	2	1	1	4	4	2	1	1	4	4	2	1
6	3	2	2	2	1	1	3	2	2	2	1	1	3	2	2	2	1	1	3
7	3	6	3	6	2	1	1	3	6	3	6	2	1	1	3	6	3	6	2
8	3	2	2	2	3	2	1	1	3	2	2	2	3	2	1	1	3	2	2
9	6	2	3	6	2	3	2	1	1	6	2	3	6	2	3	2	1	1	6
10	5	4	3	2	2	4	5	2	1	1	5	4	3	2	2	4	5	2	1
11	10	5	5	5	10	10	10	5	2	1	1	10	5	5	5	10	10	10	5
12	4	3	2	2	2	2	3	2	3	2	1	1	4	3	2	2	2	2	3
13	12	3	6	4	12	12	4	3	6	12	2	1	1	12	3	6	4	12	12
14	4	6	4	6	3	2	2	3	7	3	7	2	1	1	4	6	4	6	3
15	4	5	2	3	2	4	4	3	2	2	5	4	2	1	1	4	5	2	3
16	4	4	2	4	4	2	2	2	4	4	2	4	4	2	1	1	4	4	2
17	8	16	4	16	16	16	8	8	16	16	16	4	16	8	2	1	1	8	16
18	7	3	4	6	2	3	3	2	2	6	2	3	7	3	4	2	1	1	7
19	18	18	9	9	9	3	6	9	18	3	6	18	18	18	18	9	2	1	1
20	6	4	3	2	3	4	5	2	2	2	5	4	4	3	2	4	6	2	1

The above table shows the total number of figures required to express completely $\frac{1}{N}$ in scale *r*. Thus $\frac{1}{8} = \cdot 1\dot{6}$ (scale x.); therefore the entry under 10 and to the right of 6 is 2.

It will be noted that the numbers recur. Thus the numbers to the right of 7 are 3636211, and this group of 7 numbers is repeated indefinitely. It can be proved that for $\frac{1}{N}$ we have similarly a group of *N* numbers indefinitely repeated.

The most advantageous scale for expressing radix fractions concisely appears to be ix., followed closely by xvi. and iv. Scales formed of square numbers might be expected to prove advantageous because $r^n - 1$ has more factors when *n* is even than when *n* is odd. Scale xxv. is, in fact, better than any of those given. W. P. WORKMAN.

3. [V.] *"Italian" method of division.*

Mr. W. P. Workman reminds us of two references given by Mr. E. M. Langley in his paper on Horner's Method, printed in the 18th Annual Report of the A.I.G.T. 1892, p. 34 (note):—W. Webster's *Compendious Course of Practical Mathematics*, 1730, in the main translated from Hoste's tracts, gives the name "Italian" to the ordinary school method as well as to that combining subtraction and multiplication. O'Gorman, in his *Intuitive Calculations*, "speaks of a modification of the method in question as better than the French, German, or Italian method, but does not specify which is which." We learn, on Prof. W. H. H. Hudson's authority, that Peacock applied the term to the ordinary school method only.

8. [V.] Dr. N. Quint, of the Hague, reminds us that he has collected several mnemonics giving the value of π, in *Wiskundig Tijdschrift* (Vol. II.

pp. 169, 240, Vol. III. p. 65), of which he is a joint editor. The first quatrain is due to Lucas.

"Que j'aime à faire apprendre un nombre utile aux sages !
Immortel Archimède ! artiste ! ingénieur !
Qui de ton jugement peut priser la valeur ?
Pour moi ton problème eut de pareils avantages,"

giving the value to 30 places. Of this there is another and shorter version :

"Que j'aime à faire apprendre un nombre utile aux sages !
Glorieux Archimède ! artiste ingénieux !
Toi de qui Syracuse aime encore la mémoire !"

Of this Dr. Mackay has remarked that the rhyme is faulty, the last word contains one letter too many, while the statement as to Syracuse cherishing the memory of Archimedes must be taken as ironical (cf. *Tusculanae Disputationes*, Bk. V.).

Zacharias Dahse a native of Hamburg, and a precursor of Inaudi and the like, found the value to 200 places, of which the following verses attributed to him give 23 :

"Wie, o dies π macht ernstlich so vielen viele Müh ?
Lernt immerhin, Jünglinge, leichte Verselein
Wie so zum Beispiel dies möchte zu merken sein."

Metzdorff gives 30 places as follows :

"Dir, o Held, o alter Philosoph, Du Riesen-Genie !
Wie viel Tausende bewundern Geister himmlisch wie Du und göttlich !
Noch reiner in Aeonen wird das uns strahlen wie im lichten Morgenrot !"

Remembering that Archimedes found $3\frac{10}{71} < \pi < 3\frac{10}{70}$, a couplet of unknown origin is useful :

"How I wish I could recollect of circle round
The exact relation Archimede unwound !"

From a mysterious F.R.S. we have :

"Sir ! I send a rhyme excelling—In sacred truth and rigid spelling ;
Numerical sprites elucidate—For me the lexicon's dull weight.
If Nature gain—Not you complain—Tho' Dr. Johnson fulminate."

Or again :

"Now I know a spell unfailing—An artful charm for tasks availing,
Intricate results entailing. Not in too exacting mood,
(Poetry is pretty good).—Try the talisman. Let be
Adverse ingenuity."

The following by one of our own members, Mr. C. S. Jackson, is as witty as any, and seems as likely to be handed down to posterity.

Mnemonics usually consist of a sentence in which the number of letters in each word gives the corresponding figure of π. It is not difficult to make one up, thus :

"May I tell a story purposing to render clear the ratio circular perimeter-
3 1 4 1 5 9 2 6 5 3 5 8 9
breadth, revealing one of the problems most famous in modern days, and the
7 9 3 2 3 8 4 6 2 6 4 3 3
greatest man of science anciently known."
8 3 2 7 9 5

The story ends in nothing so the next figure is naturally 0 : or again 33 figures furnish in general a sufficiently close approximation, so the correction given by the next figure is 0.

"Solidus" is responsible for the following lucubration :—"For circumscribing a round enclosure or circle every man might remember ingenious numbers measuring one by one diameter into circle or circle upon its own diameter. Now it remains extremely close."

Dr. Mackay (*Proc. Edin. Math. Soc.*, Vol. III. p. 106) gives a French mnemonic for the reciprocal of π. Just as we we speak of '45 for 1745, so we may speak of '89 for 1789. The two French revolutions were in 1789 and 1830, and three days sufficed for the second. Hence the lines:

"Les 3 journées des 1830 on précédé 89 à l'envers," *i.e.* ·3 1830 98.

Or again:

"Les 3 jours de 1830 sont un '89 renversé."

He also notes that the simple question "Can I discover the reciprocal?" gives the value to six places, viz. 318310.

The value of e to ten places is given by:

"Tu aideras à rappeler ta quantité à beaucoup de docteurs amis";

and to 12 places from

"We proffer a mnemonic to remember a standard or neperean base value instantly."

In the same number of the *Proceedings*, is a jingle for the roots of

$$ax^2+bx+c=0.$$

"From square of b take $4ac$; square root extract and b subtract;
Divide by $2a$, you've x alway."

Or as Mr. N. D. Beatson Bell would have it:

When you have written $-b$, The double sign put down;
Then b^2-4ac, With square root mark you crown;
Beneath it all a line you trace,
Beneath which line $2a$ you place.

Dr. Mackay's amusing collection ends with:

"When rays do pass from air to glass—the value of μ is three by two.
But when they pass from air to water—the value of μ is one by three-quarter (s)!"

12. [v.] My plan is this: after a very short course of indices the boys are made to draw some such graph as that of $y=2^x$; and after calculating arithmetically $10^{\frac{1}{2}}$, $10^{\frac{1}{4}}$, and hence $10^{\frac{3}{4}}$ they draw the graph of $y=10^x$. After practice in reading it both ways they use it to make some such calculations as follows:

$$\frac{3{\cdot}371\times 4{\cdot}341}{9{\cdot}323}$$
$$=\frac{10^{\cdot528}\times 10^{\cdot638}}{10^{\cdot970}}$$
$$=10^{\cdot528+\cdot638-\cdot970}$$
$$=10^{\cdot196}=1{\cdot}157.$$

Tables of logarithms soon become intelligible. J. P. KIRKMAN.

12. [v.] *Three lessons on Indices and Logarithms.*

Age of boys 12-14. A backward class.

Question.	*Answer.*
Define x^3.	$x\times x\times x$.
Q. Define x^6.	*A.* $x\times x\times x\times x\times x\times x$.
Q. Define x^{14}.	*A.* x multiplied by itself 13 times, or 14 x's multiplied together.
Q. Define x^{17}.	*A.* 17 x's multiplied together.
QQ. [Several like the last.]	
Q. Define x^{-5}.	*A.* -5 x's multiplied together.

Q. Come to the board and do this. *A.* It can't be done, Sir.

Well, then, your definition is not a good one. We must find a better.

Q. What is the chief property of Indices? *A.* [Not forthcoming.]

Q. What is the Index Law? *A.* When multiplying powers of the same letter, add the indices to get the index of the product. [Obtained in bits and pieced together.]

Q. Give an example. *A.* $x^3 \times x^5 = x^8$.

Q. Prove this. *A.* (As usual.)

Q. Give another example. *A.* $x^4 \times x^6 = x^{10}$.

Q. Prove this. *A.* (As usual.)

Q. State the rule generally, in symbols. *A.* $x^m \times x^n = x^{m+n}$.

Q. Prove this. *A.* (As usual.)

Q. Will this do for any values of m and n? *A.* Yes.

Q. Write it, on the board, for $x^{\frac{3}{2}} \times x^4$. *A.* Can't, sir.

Then it will not do for any values of m and n.

Q. What must m and n be? *A.* Whole numbers.

Q. Try it for $x^{-2} \times x^4$. *A.* It won't work, sir.

Then it will not do for any whole numbers.

Q. What sort of numbers must m and n be? *A.* Positive whole numbers.

Q. What is the Index Law? *A.* (As before.)

Such symbols as x^{-2}, $x^{\frac{3}{2}}$, etc. are defined as symbols which obey the index law.

Q. Define x^3. *A.* 8 x's multiplied together.

Q. Define x^{-3}. *A.* (After one or two incorrect replies). A quantity which obeys the index law.

[Many more like these.]

Q. If x^{-3} is defined as a quantity which obeys the index law, how shall we find its value? *A.* By multiplying it by another power of x.

Q. What power of x shall we choose? *A.* x^3.

Q. What is $x^{-3} \times x^3$ equal to? *A.* x^{-1}.

Q. What is x^{-1} equal to? *A.* Don't know sir.

Then we must try some other power of x.

Q. (Repeat.) *A.* x^4.

Q. What is the value of $x^{-3} \times x^4$? *A.* x.

Q. What do you get on dividing each side by x^4? *A.* $\frac{x}{x^4} = \frac{1}{x^3}$.

QQ. [Drill on negative indices.]

This was followed up, similarly, with x^0, $x^{\frac{1}{2}}$, $x^{\frac{1}{3}}$, $x^{\frac{2}{3}}$, $x^{\frac{3}{4}}$, ... $x^{\frac{p}{q}}$.

This way of expressing powers of x, *i.e.* by indices, was invented by a Frenchman named René Descartes, about the year 1637. Other plans were

in use before his time. Thus a Dutch mathematician, Stevinus (about 1586), wrote $\textcircled{3}-2\textcircled{2}+3\textcircled{1}-4\textcircled{0}$ for x^3-2x^2+3x-4,

You see, the x was understood, and he simply enclosed the indices in a circle to distinguish them from other numbers.

Q. What is the value of $\textcircled{3}\times\textcircled{5}$? *A.* $\textcircled{8}$.

Q. What does this mean? *A.* $x^3\times x^5=x^8$.

Stevinus is famous as the teacher of Prince Maurice of Nassau, whose knowledge of mechanics enabled him to revolutionise the art of war and to recapture many Dutch towns from the Spaniards.

Q. What does LXX mean? *A.* 70.

Q. What does soixante-dix mean? *A.* 70. [etc.]

Just as there are many names for 70, so there are other names for indices. They are often called logarithms, the number of which they indicate a power being termed the base.

Q. What is the logarithm of x^3 to the base x? *A.* 3.

Q. Why? *A.* Because x to the power of $3=x^3$.

Q. What is the logarithm of 8 to the base 2? *A.* 3.

Q. Why? *A.* Because $2^3=8$.

[Many more like this.]

Q. What is the value of 10^0? *A.* 1, if it is defined as obeying the index law.

Q. What is the value of $\log_{10}1$? *A.* 0.

Q. Why? *A.* Because $10^0=1$.

[Many more like this.]

Q. What is the index law? *A.* (As before.)

Q. What other name do you know for an index? *A.* Logarithm.

Q. How would you get the index of the product of two powers of x? *A.* Add the indices of those powers.

Q. How would you find the logarithm of the product of two factors? *A.* Add the logarithms of the factors.

Q. If $\log_{10}2=\cdot3$ and $\log_{10}3=\cdot5$, approx., what is $\log_{10}6$? *A.* ·8, approx.

Etc., etc., *ad lib.*

Middlesbrough. A. F. VAN DER HEYDEN.

CORRESPONDENCE.

AN INDIAN MATHEMATICAL SOCIETY.

FROM Mr. V. Ramaswami Aiyar we have received the following most interesting and welcome communication, which we hope will serve as an incentive to the formation of branches of the Association in the provinces at home:—

"It may perhaps interest you to know that I have formed a Mathematical Society here. It started with twenty members, and twenty-three more have

up to date proposed to join. I shall not be surprised if our membership at the end of the year be something like 75. The annual subscription is 25 rupees (£1. 13s. 4d.). Our object just now is to have a central library from which to obtain Mathematical journals and books, for opportunities of seeing which those interested in Mathematics in India have very few facilities. I have already written for about 30 to 40 journals—English, American, French, and German. At present it is a mere book and journal club, but when the membership is sufficiently advanced we may possibly be able to start an elementary Mathematical Journal. If the Society succeeds its success will in no small measure be due to the *Mathematical Gazette*, which is the chief source of mathematical enlightenment in India. Tantalising reviews appear in the *Gazette*, but the individual has not the means of getting the text-books. We are ambitious of getting all the best books and journals and placing them within the reach of every member. At the same time we hope that Indian students may receive such a stimulus as will elicit capacity for research in the years to come.

GOOTY, *May 16th*, 1907.

MATHEMATICAL NOTES.

239. [V. 1. a.] "*Formulae*" *Problems.*

"In this connection the best assistance, to my mind, would be given by sets of simple illustrations of use of mathematics drawn from real life, so that by degrees the text-book examples would become more living and concrete."—Quoted from a letter of A. Lodge in *Math. Gazette*, May, 1907.

Formulae problems constitute one of the features of "reformed" arithmetic and algebra, and afford much better and more interesting practice in handling equations and logarithms than most of the rather dreary chapters containing scores of stereotyped examples in "x" in the text-books. The chief difficulty is to get together a fair collection of such formulae based on reality and capable of being used readily. In this connection Civil Service examinations have within recent years furnished a large number of original and easy problems, and a selection from my own collection will possibly be of some little service to fellow teachers in elementary mathematics.

Every teacher, of course, has at his command the usual formula in interest, mensuration, statics, dynamics, etc., and I need not take up any space with these.

For equation work many of the formulae can be used for finding any one of a number of data when the others are given. This will be found really excellent and interesting work for pupils.

I should like to know whether there is any collection of such problems published. Most of those given below are from Civil Service examination papers (many are to be found in D. Mair's highly suggestive and original *School Course of Mathematics*).

1. The height H of a mountain in feet is given by

$$H=49{,}000\left(\frac{R-r}{R+r}\right)\left(1+\frac{T+t}{900}\right),$$

where R, r are the observed heights of the barometer in inches, and T, t the observed temperatures at the bottom and top of the mountain respectively.

2. Calculate the value of the expression $\dfrac{b^3(0{\cdot}00081x)}{(1+at)760}$ where $b=21$, $x=730$, $t=33$, $a=\frac{1}{273}$ (an error of one per cent. is allowed).

3. Volume of a conical pier is $V=\frac{1}{12}h(a^2+ab+b^2)$, where h is the height of pier and a, b diameters of the ends; find the volume of the Forth Bridge

piers if height of pier=36 ft., diameter of top=49 ft., diameter of base =55 ft. Given also that the inside of each pier consists of 2400 c. yards of rubble, find the volume of the granite exterior in c. yards.

4. Volume of a cask in gallons is $V=\frac{L}{1059}(2D^2+E^2)$ where L is length, D middle diameter, E end diameter in inches.

5. Calculate the length of chain required for one side of a suspension bridge of 300 feet span from the following: Length of chain in feet $=2\sqrt{(\frac{1}{2}c)^2+\frac{4}{3}V^2}$ where C=the span in feet, and $V=\frac{1}{13}C$.

6. If a water pipe is L yards long, d inches in diameter, and one end is H feet higher than the other, then $\sqrt{\frac{(3d)^5\times H}{L}}$ gallons of water will flow through the pipe in a minute.

If 5400 gallons flow through a pipe $\frac{3}{4}$ mile long, 4 inches in diameter, find the difference in heights of the two ends to the nearest foot.

7. Tonnage of a vessel is $\frac{(L-\frac{3}{5}B)\times B\times\frac{1}{2}B}{94}$; where L and B are length and breadth of vessel measured in feet.

8. The area of the opening under the archway of a bridge is sometimes calculated from the formula $\frac{4V}{3}\times\sqrt{(0{\cdot}626\,V)^2+C^2}$, where V is the height of the arch and C half the distance between the ends. A figure can be given to scale for this question.

9. The velocity of water in a rectangular mill-stream whose breadth is A feet and depth D feet is $\frac{520\times R}{0{\cdot}516+\sqrt{R}}$ feet per min. when $R=\frac{AD}{2D+A}$. Calculate the amount of water which would pass along such a stream when $A=4$ and $D=3$, and compare it with the amount when $A=4$ and $D=2$.

10. An estate brings in a rental of $£m\times A$ to begin with; but at the end of every q years the rental is diminished by $£A$. The present value of the estate is then found to be $\frac{A[mR^{nq+q}-(m+1)R^{nq}-2R^q+1]}{R^{nq}(R^q-1)(R-1)}$ where $R=1+\frac{3{\cdot}5}{100}$, rental=£2500, which is diminished at the end of every seven years by £250.

11. It has been found that if P is the H.P. wasted in air friction when a disc d feet in diameter is revolving at n revolutions per minute $P=cd^{5\cdot5}n^{3\cdot5}$ where c is a constant. If $P=0{\cdot}1$ when $d=4$ and $n=500$, find the value of c. Hence find the H.P. wasted when a disc 45 inches in diameter revolves 10 times a second.

12. The amount of water which can be carried off by a circular drain pipe of diameter D feet is $\frac{A\times D^3}{C+\sqrt{D}}$ c. ft. per min. where $A=454{\cdot}3$ and $C=1{\cdot}005$. Find, with as little work as possible, whether a drain pipe 21 inches in diameter will suffice to carry off 1000 c. ft. per minute.

Find in m.p.h. the average speed of the water in the pipe, using $0{\cdot}785\times D^2$ for section of pipe.

W. WATSON.

240. [K. v. d.] *Linkage Problem.*

(i) If a given triangle PAB moves in one plane, with its angles A, B on the circumferences of two given circles (centres C, D) respectively, then (a) the nodes of the locus of P will be the points where it meets a certain circle through C, D—that circumscribing the triangle CXD similar to APD (in the same sense); and (b) the two branches which intersect at such a

node will touch each other if, and only if, either of them touches the "nodal circle."

(ii) In the symmetrical case, where $PA=PB=l$, $CA=DB=r$, show that the nodes will be (a) the middle point of the arc CXD, and (b) two other points on the circle (diameter d) at a distance from $CD=\frac{l^2-r^2}{d}$. Find further the condition that these two nodes may be cusps.

(iii) Examine the special case where $CA=DB$ and $CD=AB$; and account for the existence, in this case, of additional nodes, *not lying on the nodal circle.*

Solution.

(i) (a) If two positions of P coincide, corresponding to positions AB, $A'B'$, of the base, we have $PA=PA'$, $PB=PB'$, and PC, PD bisect the equal angles APA', BPB'. Therefore the angle CPD=the angle APB; showing that P lies on the circle in question.

[It might however in some cases lie (on this circle) on the other side of CD.]

(b) Let Q be the point on the nodal circle diametrically opposite to P. The instantaneous centres of rotations O, O', for the two positions of PAB, will be the intersections of CA, DB; CA', DB', respectively. Thus O, O', P, Q are the intersections of corresponding rays of two harmonic pencils $C\{AA'PQ\}$, $D\{BB'PQ\}$. It follows that O, O' will be in the same straight line with P (the condition that the two branches may touch), only when collinear with Q, *i.e.* when the branches touch the nodal circle.

[This is quite distinct from the conditions for a cusp, when it will be found that O, O' coincide with P.]

(ii) (a) In the symmetrical case, we have $PAA'\equiv PBB'$, $CAA'\equiv DBB'$, and therefore $PM=PN$, $CM=DN$; where M, N are the intersections of PC, PD with AA', BB' respectively. Then either $PC=PD$ *each* being the sum (or difference) of these segments; or, as in the figure, one is the sum and one the difference, and we have $PC.PD$

$$=(PM+MC)(PM-MC)=PM^2-MC^2=PA^2-AC^2=l^2-r^2.$$

Therefore $l^2-r^2=d\times$ the perpendicular distance of P from CD.

[This may be negative.]

(b) Such a node will become a cusp if AA', BB' coincide, so that PC, $PD=l\pm r$. Putting $AB=2a$, $CD=2b$, we then have

$$\frac{(l+r)^2+(l-r)^2-(2b)^2}{2(l^2-r^2)}=\cos P=1-2\frac{a^2}{l^2};\ \text{or } l^2(l^2+r^2-2b^2)=(l^2-r^2)(l^2-2a^2);$$

i.e. $l^2(r^2+a^2-b^2)=a^2r^2$, the condition for cusps.

(iii) If $AB=CD$, $AC=BD$, one possible motion will be that in which AB keeps parallel to itself throughout, when P will describe a circle of radius r; but there will be another part of the locus which will, in general, cut the circular part in four points (besides having a central node). Only two of these will lie on the nodal circle. The other two will be found to lie at the extremities of the diameter parallel to CD, of the circular locus; and they correspond each to a position of AB, (in line with CD), *from which there is two-way motion possible*, not to *two different positions of AB for which P is the same.* The latter is the case hitherto considered; the other is exceptional.

PERCY J. HEAWOOD.

241. [A. 3. d.] *The limits of the positive roots of* $x^3-qx+r=0$.

In the equation $x^3-qx+r=0$, where r is positive, there is certainly one real negative root. The remaining roots may be either both imaginary or both real and positive.

In the latter case it is proposed to show that the two positive roots must lie between O and $q^2/4r$.

If α, β, $-(\alpha+\beta)$ are the the three roots (α, β both positive) the given equation is equivalent to

$$\{x+(\alpha+\beta)\}\{x^2-(\alpha+\beta)+\alpha\beta\}=0,$$

or
$$x^3-(\alpha^2+\alpha\beta+\beta^2)x+\alpha\beta(\alpha+\beta)=0;$$

so that
$$q=\alpha^2+\alpha\beta+\beta^2,\quad r=\alpha\beta(\alpha+\beta).$$

Now $(\alpha^2+\alpha\beta+\beta^2)^2/4\alpha\beta(\alpha+\beta)$ is $>\alpha$ and also $>\beta$.

For $(\alpha^2+\alpha\beta+\beta^2)^2-4\alpha^2\beta(\alpha+\beta)=(\alpha^2-\alpha\beta-\beta^2)^2$, etc.

Therefore $q^2/4r$ is greater than either α or β.

[It may be noticed that if we substitute $q^2/4r$ for x in x^3-qx+r we get $[(q^3-8r^2)/8r]^2/r$ which is of the same sign as r, so that *a priori* both α and β lie either between O and $q^2/4r$ or between $q^2/4r$ and $+\infty$.]

The above inequality applied to the quadratic $x^2-Px+Q=0$ gives $(P^2-Q)^2/4PQ$ as a superior or an inferior limit according as P, Q are of the same or different sign (the roots being of course *real*). R. F. DAVIS.

242. [T. 3. b.] *Convenient rules for Elementary Calculations as to Thin Lenses.*

The *divergence* of a pencil of rays with reference to a thin lens is defined as the reciprocal of the distance of its vertex from the centre of the lens. If the rays are convergent, their *divergence* is reckoned negative.

The *divergive power* of a lens is the divergence it imparts to parallel rays passing through it, and is equal to the reciprocal of the focal length of the lens, taken negatively if the lens be *convergive*. Its metric unit is the *diopter*.

RULE for finding the position of the image of a point produced by a thin lens:

Divergence after refraction = divergence before + divergive power of lens,

or
$$d'=d+p.$$

Here, of course, attention must be paid to the *signs* of the quantities dealt with.

Take, for instance, a convergive lens whose focal length is 6 inches and whose divergive power is therefore $-\frac{1}{6}$. If a point be placed 8 inches in front of it, where is the image? $d'=\frac{1}{8}-\frac{1}{6}=-\frac{1}{24}$. Thus the rays have become *convergent* and meet at a point 24 inches *beyond* the lens.

Again, if the point be brought to within 4 inches of the lens we have $d'=\frac{1}{4}-\frac{1}{6}=\frac{1}{12}$. Thus the rays remain *divergent* and the image is behind the object, at distance 12 inches from the lens.

RULE for calculating the divergive power of a thin lens when its dimensions and material are given.

Divergive power = refractivity × (sum of concavities),

or
$$p=(\mu-1)\left(\pm\frac{1}{r}\pm\frac{1}{r'}\right).$$

Here the *refractivity* is defined as the excess of the refractive index of the glass over that of air, and by concavity is meant the spherical curvature of a face of the lens, taken as positive if it is concave (negative if convex).

The advantage of the rule in words over that in symbols is that there is no ± sign to remember the interpretation of. We may note that of the two factors of divergive power, the former is a function of its *material*, the latter of its *form*. Again, the divergive power of a combination of two thin lenses placed in contact is the algebraic sum of their divergive powers.

Rules similar to the above might be stated, substituting *convergence*, *convergive power*, and *convexity* for *divergence*, *divergive power*, and *concavity*, but it is simpler to keep to one set of rules.

What precedes is the gist of the method taught by Lord Kelvin when Professor of Natural Philosophy in Glasgow; to a great extent it was anticipated by Coddington in his *Treatise on the Reflexion and Refraction of Light*, Cambridge, 1829. An account of Kelvin's method was published by Dr. Magnus Maclean in the *Phil. Mag.*, Nov., 1889. Another, with an extension to the case of refraction at a single spherical surface, was given by me in the *Proceedings of the Royal Philosophical Society of Glasgow*, April, 1903.

R. F. MUIRHEAD.

REVIEWS.

Theoretical Mechanics. By J. H. JEANS, F.R.S., Professor of Applied Mathematics in Princeton University. Pp. vii, 364. Ginn & Co., Boston and London. Price 10s. 6d.

All teachers of elementary mechanics rejoice when one who has shewn his powers as an original investigator, as Professor Jeans has done, devotes himself to producing an elementary exposition of the science.

From such an author we seek for guidance on points of difficulty, for new light on familiar matters, for perspective and a mode of treatment in which topics shall appear in their due order and magnitude: nor is our search in vain.

The object of the book may be indicated by an extract from the preface:

"All students will best begin the subject by trying to acquire a firm grasp of physical principles, leaving aside at first mathematical developments and practical applications, except in so far as these contribute to the elucidation of the fundamental physical principles."

Among the many features of novelty and interest in Professor Jeans' treatise may be mentioned a striking introductory paragraph on the uniformity of nature as the basis of science; many original and ingenious examples, the use of the calculus in close connection with the work and yet in such a manner that the cardinal principles are presented without the calculus; the mode of obtaining the principle of the transmissibility of force; the reconciliation of the relativity of velocity, and the dependence of v on the frame of reference employed, with the principle of work and the formula $\frac{1}{2}mv^2$ for kinetic energy; and especially the final chapter on generalised coordinates. As is inevitable where so much of the treatment is novel there are a few of those slips which are so invisible in proofsheets and so obvious in a published book, and a few places where the ordinary student will require additional stepping stones.

Thus at page 72 a formula is obtained for the advantage obtained by using a wheeled car instead of a sledge, and it is remarked that the formula shews that the wheel still possesses an advantage even if the coefficient of friction at the axle is infinite. But if a wheel is locked on the axle it can only slide on the ground.

Again, the numerical illustration of the top in section 257 is vitiated by taking B as the moment of inertia about an axis through the C.G. instead of about a parallel axis through the fixed point.

Again, would not many students object to *writing down* $B\sin^2\theta\dot{\phi}$ as that part of the moment of momentum of a top about a vertical axis which is due to the motion of the axis.

Why is the coefficient B the moment of inertia about a principal axis through the point of contact other than the axis of symmetry?

The proposition on which the result depends, and which is not explicitly stated by Professor Jeans may be indicated thus:—I_1, I_2, I_3 being moments of inertia about principal axes 1, 2, 3; I the moment of inertia about an axis whose direction cosines are l_1, l_2, l_3, and the angular velocity about the axis l_1, l_2, l_3 being Ω, vectorially compounded of Ω_1, Ω_2, Ω_3, we have for the moment of momentum about l_1, l_2, l_3.

$$I\Omega=[I_1l_1^2+I_2l_2^2+I_3l_3^2]\Omega$$
$$=I_1\Omega_1l_1+I_2\Omega_2l_2+I_3\Omega_3l_3,$$

so that the moment of momentum is vectorially compounded of $I_1\Omega_1$, $I_2\Omega_2$, and $I_3\Omega_3$.

In the case of the top, with the usual notation

$$I_1 = A \qquad I_2 = I_3 = B,$$

$$l_1 = \cos\theta \qquad l_2 = \cos\left(\frac{\pi}{2} + \theta\right) \qquad l_3 = 0,$$

$$\Omega_1 = \omega \qquad \Omega_2 = \sin\theta\dot{\phi} \qquad \Omega_3 = -\dot{\theta}.$$

The great value of the book is not however affected by such comparatively small matters as these, and the work will take its place as one of the most useful to which a teacher of elementary rigid dynamics can refer for aid in presenting this difficult subject. C. S. JACKSON.

Tablettes des Cotes relatives à la base 20580 des facteurs premier d'un nombre inférieur à N et non divisible par 2, 3, 5, ou 7. Par GASTON TARRY. 2 pp. Text, 6 pp. Tables. l. 8vo. Paris, Gauthier-Villars, 1906.

These tables form a very ingenious and highly compact apparatus for detecting the divisors $\geqslant 313$ of any number N. Taking as "base" $b = 2.3.5.7^2 = 20580$, the "character" (cote) of a number k relative to a prime modulus p (>7) is defined as the *least* residue ($\pm$) of ka to mod. p, where $ab = +1$ (mod. p). Two tables of "characters" are provided. The main table is that of the characters of r to mod. p, where $r < 2.3.5.7 = 210$, and covers two large 8vo pages. The second table gives the characters of $210q$ to mod. p, where $q \geqslant 49$, covers four such pages, and is pierced with long narrow slits of the width of a column. Each prime modulus p has a line to itself; thus each table has 60 lines on a page, *i.e.* one line for each of the 60 primes from 11 to 313.

The number N to be factorised is to be expressed in the form $N = mb \pm (210q + r)$, in such a way that the residue $(210q + r)$ shall be $< \frac{1}{2}b$; here r is $+$ and < 210. Taking q, r as arguments (these are the numbers at the heads of the columns in the q- and r-tables), the q-table is to be placed over the r-table with the q-column alongside the r-column, the figures in which will be visible through the slits. Denoting the "characters" of $210q$ and r by the letters Q, R, (these are the entries in the q- and r-columns) the adjacent entries Q, R are to be compared throughout the whole length of the q- and r-columns. The points to be examined are (under certain rules as to signs of Q, R, m).

1. If $Q - R = \pm m$; 2. If $Q + R = m$; 3. If $Q + R + m = p$.

In any of these cases p must be a divisor of N, and not otherwise. This arrangement is certainly very ingenious: the actual trial division by a divisor (p) is here replaced by an examination of the three simple conditions above. The original expression of the number N in the form $N = mb \pm (210q + r)$ is the *only* computation required, when $m \geqslant 5$, *i.e.* when $N \geqslant 113030$. With larger numbers (N), some of the prime moduli (p) will be $< 2m$: in such cases m is to be replaced by its least residue to mod. p, in the three conditions above.

As at present prepared the tables extend only to 60 lines (for 60 primes), *i.e.* up to $p = 313$: to be of any great practical use they would require to be greatly extended. Their chief merit would appear to be in their great compactness, as they are of course not so easy to use as the ordinary tables. Their extension will evidently involve great labour, and will therefore be costly: also, there will be great risk of error as the "characters" are quite artificial numbers. The sheets of the q-tables being pierced with five long slits, running the whole length of the printed matter, are somewhat delicate, and will need care in handling: this is a decided practical drawback. ALLAN CUNNINGHAM, Lt.-Col., R.E.

Geodasie. PROF. DR. A. GALLE. Leipzig. G. J. Goschen. 1907. Pp. vii, 284. Price, 8 marks.

The mathematical principles of topographical surveying as distinguished from the details of the construction and use of topographical instruments are carefully set forth by Professor Galle. Special attention is given to the "three point problem" to the application of photography to surveying, and to the effect on final results of errors in observed quantities. A teacher desirous of making trigonometry "practical" would find much to interest him in this volume of the "Sammlung Schubert." C. S. J.

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